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SUPERSYMMETRIES AND RECURSION OPERATORS FOR $N=2$ SUPERSYMMETRIC KDV-EQUATION

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1 Introduction

In this lecture we shall discuss the supersymmetric extensions of the classical KdV equation

$$u_t = -u_{xxx} + 6uu_x \tag{1}$$

with two odd variables, the situation $N=2$. The construction of such supersymmetric systems runs along similar lines as for the supersymmetric extension of the classical nonlinear Schrödinger equation. For additional references see also [1, 2, 3, 5].

The extension is obtained by considering two odd (pseudo) total derivative operators D_1 and D_2 given by

$$D_1 = \partial_{\theta_1} + \theta_1 D_x, \quad D_2 = \partial_{\theta_2} + \theta_2 D_x, \tag{2}$$

where θ_1, θ_2 are two odd parameters. Obviously, these operators satisfy the relations $D_1^2 = D_2^2 = D_x$ and $[D_1, D_2] = 0$.

The $N=2$ supersymmetric extension of the KdV equation is obtained by taking an even homogeneous field Φ

$$\Phi = w + \theta_1\psi + \theta_2\varphi + \theta_2\theta_1 u \tag{3}$$

with degrees $\deg(\Phi) = 1$, $\deg(u) = 2$, $\deg(w) = 1$, $\deg(\varphi) = \deg(\psi) = 3/2$, $\deg(\theta_1) = \deg(\theta_2) = -1/2$, and considering the most general evolution equation for Φ, which reduces to the KdV equation in the absence of the odd variables φ, ψ.

Proceeding in this way, we arrive at the system

$$\Phi_t = D_x\left(-D_x^2\Phi + 3\Phi D_1D_2\Phi + \frac{1}{2}(a-1)D_1D_2\Phi^2 + a\Phi^3\right). \tag{4}$$

Rewriting this system in components, we arrive at a system of partial differential equations for the two even variables u, w and the two odd variables φ, ψ, i.e.,

$$\begin{aligned}
u_t &= -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 - 3aw_1w_2 - (a+2)ww_3 + 3au_1w^2 \\
&\quad + 6auww_1 + 6aw_1\psi\varphi + 6aw\psi_1\varphi + 6aw\psi\varphi_1, \\
\varphi_t &= -\varphi_3 + 3u_1\varphi + 3u\varphi_1 + 6aww_1\varphi + 3aw^2\varphi_1 - (a+2)w_1\psi_1 \\
&\quad - (a+2)w\psi_2 - (a-1)w_2\psi - (a-1)w_1\psi_1, \\
\psi_t &= -\psi_3 + 3u_1\psi + 3u\psi_1 + 6aww_1\psi + 3aw^2\psi_1 + (a+2)w_1\varphi_1 \\
&\quad + (a+2)w\varphi_2 + (a-1)w_2\varphi + (a-1)w_1\varphi_1, \\
w_t &= -w_3 + 3aw^2w_1 + (a+2)u_1w + (a+2)uw_1 + (a-1)\psi_1\varphi \\
&\quad + (a-1)\psi\varphi_1.
\end{aligned} \tag{5}$$

It has been demonstrated by several authors [2, 1] that the interesting equations from the point of view of complete integrability are the special cases $a = -2, 1, 4$.

First we discuss the case $a = -2$. We shall present results for the construction of local and nonlocal conservation laws, nonlocal symmetries and finally present the recursion operator for symmetries.
It should be stressed that all constructions and computations are carried through along the theoretical lines laid down in [4, 5] A similar presentation is chosen for the case $a = 4$, and finally for the most intriguing case $a = 1$,the results of which are given in [5].
The structure is extremely complicated in this last case. The reason for the complexity is strongly related to the appearance of nonlocal variables of degree 0, which play an essential role in the construction of symmetries , conservation laws, recursion symmetries.

2 Case $a = -2$

In this subsection we discuss the case $a = -2$, which leads to the following system of partial differential equations

$$
\begin{aligned}
u_t &= -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 + 6w_1w_2 - 6u_1w^2 - 12uww_1 \\
&\quad - 12w_1\psi\varphi - 12w\psi_1\varphi - 12w\psi\varphi_1, \\
\varphi_t &= -\varphi_3 + 3u_1\varphi + 3u\varphi_1 - 12ww_1\varphi - 6w^2\varphi_1 + 3w_2\psi + 3w_1\psi_1, \\
\psi_t &= -\psi_3 + 3u_1\psi + 3u\psi_1 - 12ww_1\psi - 6w^2\psi_1 - 3w_2\varphi - 3w_1\varphi_1, \\
w_t &= -w_3 - 6w^2w_1 - 3\psi_1\varphi - 3\psi\varphi_1.
\end{aligned} \tag{6}
$$

The results obtained in this case for conservation laws, higher symmetries and deformations or recursion operator will be presented in subsequent subsections.

2.1 Conservation laws

For the even conservation laws and the associated even nonlocal variables we obtained the following results.

1. Nonlocal variables $p_{0,1}$ and $p_{0,2}$ of degree 0 defined by

$$
\begin{aligned}
(p_{0,1})_x &= w, \\
(p_{0,1})_t &= 3\varphi\psi - 2w^3 - w_2; \\
(p_{0,2})_x &= p_{1,1}, \\
(p_{0,2})_t &= 12p_{3,1} - u_1 + 3ww_1
\end{aligned} \tag{7}
$$

(see the definition of $p_{1,1}$ and $p_{3,1}$ below).

2. Nonlocal variables $p_{1,1}, p_{1,2}, p_{1,3}, p_{1,4}$ of degree 1 defined by the relations

$$
\begin{aligned}
(p_{1,1})_x &= u, \\
(p_{1,1})_t &= -3\psi\psi_1 - 3\varphi\varphi_1 + 12\varphi\psi w + 3u^2 - 6uw^2 - u_2 + 3w_1^2; \\
(p_{1,2})_x &= \psi\overline{q}_{\frac{1}{2}} - \varphi q_{\frac{1}{2}}, \\
(p_{1,2})_t &= -\psi_2\overline{q}_{\frac{1}{2}} + \varphi_2 q_{\frac{1}{2}} + 3\psi\overline{q}_{\frac{1}{2}}u \\
&\quad - 6\psi\overline{q}_{\frac{1}{2}}w^2 - 3\psi q_{\frac{1}{2}}w_1 - 2\psi\psi_1 - 3\varphi\overline{q}_{\frac{1}{2}}w_1 - 3\varphi q_{\frac{1}{2}}u + 6\varphi q_{\frac{1}{2}}w^2 + 2\varphi\varphi_1; \\
(p_{1,3})_x &= \psi q_{\frac{1}{2}}, \\
(p_{1,3})_t &= -\psi_2 q_{\frac{1}{2}} + 3\psi q_{\frac{1}{2}}u - 6\psi q_{\frac{1}{2}}w^2 + \varphi_1\psi - 3\varphi q_{\frac{1}{2}}w_1 - \varphi\psi_1; \\
(p_{1,4})_x &= \varphi q_{\frac{1}{2}} + w^2, \\
(p_{1,4})_t &= -\varphi_2 q_{\frac{1}{2}} + 3\psi q_{\frac{1}{2}}w_1 + 3\varphi q_{\frac{1}{2}}u - 6\varphi q_{\frac{1}{2}}w^2 \\
&\quad - 2\varphi\varphi_1 + 6\varphi\psi w - 3w^4 - 2ww_2 + w_1^2
\end{aligned}
\tag{8}
$$

(the variables $q_{\frac{1}{2}}$ and $\overline{q}_{\frac{1}{2}}$ are defined below).

3. Nonlocal variable $p_{2,1}$ of degree 2 defined by (omitting $(p_{2,1})_t$, for simplicity)

$$(p_{2,1})_x = q_{\frac{1}{2}}\overline{q}_{\frac{1}{2}}u + \psi_1 q_{\frac{1}{2}} + \psi\overline{q}_{\frac{1}{2}}w + \varphi q_{\frac{1}{2}}w. \tag{9}$$

4. Finally, the variable $p_{3,1}$ of degree 3 defined by (omitting $(p_{3,1})_t$, for simplicity)

$$(p_{3,1})_x = \frac{1}{4}(-\psi\psi_1 - \varphi\varphi_1 + 4\varphi\psi w + u^2 - 2uw^2 - ww_2). \tag{10}$$

Remark
It should be noted that the first lower index refers to the degree of the object (in this case the nonlocal variable), while the second lower index is referring to the numbering of the objects of that specific degree. The number of nonlocal variables of degree 3 is 4, since this number is the same as for nonlocal variables of degree 1, cf. (8). This total number will arise after introduction of these nonlocal variables and computation of the conservation laws and the associated nonlocal variables in this augmented setting. These conservation laws and their associated nonlocal variables are of a higher nonlocality. We

shall not pursue this further here, because the number of nonlocal variables found will turn out to be sufficient to compute the deformation of the system of equations (6), or equivalently the construction of the recursion operator for symmetries.
For the odd conservation laws and the associated odd nonlocal variables we derived the following results.

1. At degree 1/2 we computed the variables $q_{\frac{1}{2}}$ and $\overline{q}_{\frac{1}{2}}$ defined by

$$\begin{aligned}(q_{\frac{1}{2}})_x &= \varphi,\\ (q_{\frac{1}{2}})_t &= -\varphi_2 + 3\psi w_1 + 3\varphi u - 6\varphi w^2;\\ (\overline{q}_{\frac{1}{2}})_x &= \psi,\\ (\overline{q}_{\frac{1}{2}})_t &= -\psi_2 + 3\psi u - 6\psi w^2 - 3\varphi w_1.\end{aligned} \tag{11}$$

2. At degree 3/2 we have the variables $q_{\frac{3}{2}}$ and $\overline{q}_{\frac{3}{2}}$ defined by

$$\begin{aligned}(q_{\frac{3}{2}})_x &= \overline{q}_{\frac{1}{2}}u - \varphi w,\\ (q_{\frac{3}{2}})_t &= 3\overline{q}_{\frac{1}{2}}u^2 - 6\overline{q}_{\frac{1}{2}}uw^2 - \overline{q}_{\frac{1}{2}}u_2 + 3\overline{q}_{\frac{1}{2}}w_1^2 + \varphi_2 w - \psi_1 u - \varphi_1 w_1 - 3\psi\psi_1\overline{q}_{\frac{1}{2}}\\ &\quad + \psi u_1 - 3\psi w w_1 - 3\varphi\varphi_1\overline{q}_{\frac{1}{2}} + 12\varphi\psi\overline{q}_{\frac{1}{2}}w - 3\varphi u w + 6\varphi w^3 + \varphi w_2;\\ (\overline{q}_{\frac{3}{2}})_x &= -(q_{\frac{1}{2}}u + \psi w),\\ (\overline{q}_{\frac{3}{2}})_t &= -3q_{\frac{1}{2}}u^2 + 6q_{\frac{1}{2}}uw^2 + q_{\frac{1}{2}}u_2 - 3q_{\frac{1}{2}}w_1^2 + \psi_2 w - \psi_1 w_1 + \varphi_1 u + 3\psi\psi_1 q_{\frac{1}{2}}\\ &\quad - 3\psi u w + 6\psi w^3 + \psi w_2 + 3\varphi\varphi_1 q_{\frac{1}{2}} - 12\varphi\psi q_{\frac{1}{2}}w - \varphi u_1 + 3\varphi w w_1.\end{aligned} \tag{12}$$

3. Finally, at degree 5/2 we obtained $q_{\frac{5}{2}}$ and $\overline{q}_{\frac{5}{2}}$ defined by the relations

$$\begin{aligned}(q_{\frac{5}{2}})_x &= \overline{q}_{\frac{1}{2}}p_{1,1}u + 3\overline{q}_{\frac{1}{2}}ww_1 + \varphi_1 w + \psi u - \varphi p_{1,1}w,\\ (\overline{q}_{\frac{5}{2}})_x &= -q_{\frac{1}{2}}p_{1,1}u + q_{\frac{1}{2}}u_1 - 3q_{\frac{1}{2}}ww_1 + \psi_1 w - \psi p_{1,1}w.\end{aligned} \tag{13}$$

Thus the entire nonlocal setting comprises the following 14 nonlocal variables:

$$
\begin{array}{ll}
p_{0,1},\ p_{0,2} & \text{of degree } 0, \\
p_{1,1},\ p_{1,2},\ p_{1,3},\ p_{1,4} & \text{of degree } 1, \\
p_{2,1} & \text{of degree } 2, \\
p_{3,1} & \text{of degree } 3, \\
q_{\frac{1}{2}},\ \overline{q}_{\frac{1}{2}} & \text{of degree } \frac{1}{2}, \\
q_{\frac{3}{2}},\ \overline{q}_{\frac{3}{2}} & \text{of degree } \frac{3}{2}, \\
q_{\frac{5}{2}},\ \overline{q}_{\frac{5}{2}} & \text{of degree } \frac{5}{2}.
\end{array} \tag{14}
$$

In the next subsections the augmented system of equations associated to the local and the nonlocal variables denoted above will be considered in computing higher and nonlocal symmetries and the recursion operator.

2.2 Higher and nonlocal symmetries

In this subsection, we present results for higher and nonlocal symmetries for the $N = 2$ supersymmetric extension of KdV equation (6),

$$
Y = Y^u \frac{\partial}{\partial u} + Y^w \frac{\partial}{\partial w} + Y^\varphi \frac{\partial}{\partial \varphi} + Y^\psi \frac{\partial}{\partial \psi} + \dots
$$

We obtained the following odd symmetries, just giving here the components of their generating functions,

$$
\begin{array}{ll}
Y^u_{\frac{1}{2},1} = \psi_1, & Y^u_{\frac{1}{2},2} = \varphi_1, \\
Y^w_{\frac{1}{2},1} = -\varphi, & Y^u_{\frac{1}{2},2} = \varphi_1, \\
Y^\varphi_{\frac{1}{2},1} = -w_1, & Y^\varphi_{\frac{1}{2},2} = u, \\
Y^\psi_{\frac{1}{2},1} = u; & Y^\psi_{\frac{1}{2},2} = w_1
\end{array} \tag{15}
$$

and $Y_{\frac{3}{2},1}, Y_{\frac{3}{2},2}$ whose representation is given in [5]. We also obtained the following even symmetries:

$$
\begin{aligned}
Y^{u}_{0,1} &= 0,\\
Y^{w}_{0,1} &= 0,\\
Y^{\varphi}_{0,1} &= \psi,\\
Y^{\psi}_{0,1} &= -\varphi;\\[6pt]
Y^{u}_{1,1} &= u_1,\\
Y^{w}_{1,1} &= w_1,\\
Y^{\varphi}_{1,1} &= \varphi_1,\\
Y^{\psi}_{1,1} &= \psi_1;\\[6pt]
Y^{u}_{1,2} &= \varphi_1 q_{\frac{1}{2}} + 2ww_1,\\
Y^{w}_{1,2} &= \psi q_{\frac{1}{2}} + w_1,\\
Y^{\varphi}_{1,2} &= -q_{\frac{1}{2}}u + \varphi_1 - \psi w,\\
Y^{\psi}_{1,2} &= -q_{\frac{1}{2}}w_1 - \varphi w;\\[6pt]
Y^{u}_{1,3} &= \psi_1 \overline{q}_{\frac{1}{2}} - \varphi_1 q_{\frac{1}{2}},\\
Y^{w}_{1,3} &= -\psi q_{\frac{1}{2}} - \varphi \overline{q}_{\frac{1}{2}},\\
Y^{\varphi}_{1,3} &= \overline{q}_{\frac{1}{2}}w_1 + q_{\frac{1}{2}}u - \varphi_1 + 2\psi w,\\
Y^{\psi}_{1,3} &= -\overline{q}_{\frac{1}{2}}u + q_{\frac{1}{2}}w_1 + \psi_1 + 2\varphi w;\\[6pt]
Y^{u}_{1,4} &= \psi_1 q_{\frac{1}{2}} + \varphi_1 \overline{q}_{\frac{1}{2}},\\
Y^{w}_{1,4} &= \psi \overline{q}_{\frac{1}{2}} - \varphi q_{\frac{1}{2}},\\
Y^{\varphi}_{1,4} &= -\overline{q}_{\frac{1}{2}}u + q_{\frac{1}{2}}w_1 + \psi_1 + 2\varphi w,\\
Y^{\psi}_{1,4} &= -\overline{q}_{\frac{1}{2}}w_1 - q_{\frac{1}{2}}u + \varphi_1 - 2\psi w.
\end{aligned}
\tag{16}
$$

Moreover there is a symmetry of degree 2 , $Y_{2,1}$ whose representation is given in [5].

2.3 Recursion operator

Here we present the recursion operator $\mathcal{R}$ for symmetries for this case obtained as a higher symmetry in the Cartan covering of the augmented system of equations (14). The result is

$$\mathcal{R} = R^u \frac{\partial}{\partial u} + R^w \frac{\partial}{\partial w} + R^\varphi \frac{\partial}{\partial \varphi} + R^\psi \frac{\partial}{\partial \psi} + \dots, \tag{17}$$

where the components R^u, R^w, R^φ, R^ψ are given by

$$\begin{aligned}
R_u &= \omega_{u_2} + \omega_u(-4u + 4w^2) \\
&+ \omega_{w_1}(-4w_1) + \omega_w(8uw - 2w_2 - 6\varphi\psi) \\
&+ \omega_{\varphi_1}(-2\varphi) + \omega_\varphi(\varphi_1 - 8\psi w) + \omega_{\psi_1}(-2\psi) + \omega_\psi(\psi_1 + 8\varphi w) \\
&+ \omega_{q_{\frac{1}{2}}}(\varphi_2 - 3\psi_1 w - 3\psi w_1 - \varphi u - q_{\frac{1}{2}} u_1) \\
&+ \omega_{\overline{q}_{\frac{1}{2}}}(\psi_2 + 3\varphi_1 w + 3\varphi w_1 - \psi u - \overline{q}_{\frac{1}{2}} u_1) \\
&+ \omega_{q_{\frac{3}{2}}}(\psi_1) + \omega_{\overline{q}_{\frac{3}{2}}}(-\varphi_1) + \omega_{p_{1,4}}(2u_1) + \omega_{p_{1,2}}(u_1) \\
&+ \omega_{p_{1,1}}(-2u_1 + 4ww_1 + \varphi_1 q_{\frac{1}{2}} + \psi_1 \overline{q}_{\frac{1}{2}}), \\
R_w &= \omega_{w_2} + \omega_w(4w^2) + \omega_\varphi(-2\psi) + \omega_\psi(2\varphi) \\
&+ \omega_{q_{\frac{1}{2}}}(-\psi_1 - \varphi w - q_{\frac{1}{2}} w_1) + \omega_{\overline{q}_{\frac{1}{2}}}(\varphi_1 - \psi w - \overline{q}_{\frac{1}{2}} w_1) \\
&+ \omega_{q_{\frac{3}{2}}}(-\varphi) + \omega_{\overline{q}_{\frac{3}{2}}}(-\psi) + \omega_{p_{1,4}}(2w_1) + \omega_{p_{1,2}}(w_1) \\
&+ \omega_{p_{1,1}}(\psi q_{\frac{1}{2}} - \varphi \overline{q}_{\frac{1}{2}}), \\
R_\varphi &= \omega_u(-2\varphi) + \omega_{w_1}(-2\psi) + \omega_w(-\psi_1 + 8\varphi w) \\
&+ \omega_{\varphi_2} + \omega_\varphi(-2u + 4w^2) + \omega_\psi(-2w_1) \\
&+ \omega_{q_{\frac{1}{2}}}(-u_1 + 3ww_1 + \varphi_1 q_{\frac{1}{2}}) \\
&+ \omega_{\overline{q}_{\frac{1}{2}}}(-uw - w_2 + 2\varphi\psi + \varphi_1 \overline{q}_{\frac{1}{2}}) \\
&+ \omega_{q_{\frac{3}{2}}}(-w_1) + \omega_{\overline{q}_{\frac{3}{2}}}(-u) + \omega_{p_{1,4}}(2\varphi_1) + \omega_{p_{1,2}}(\varphi_1) \\
&+ \omega_{p_{1,1}}(-\varphi_1 - q_{\frac{1}{2}} u + \overline{q}_{\frac{1}{2}} w_1),
\end{aligned} \tag{18}$$

$$\begin{aligned}R_\psi &= \omega_u(-2\psi) + \omega_{w_1}(2\varphi) + \omega_w(\varphi_1 + 8\psi w)\\ &+ \omega_\varphi(2w_1) + \omega_{\psi_2} + \omega_\psi(-2u + 4w^2)\\ &+ \omega_{q_{\frac{1}{2}}}(uw + w_2 - 2\varphi\psi + \psi_1 q_{\frac{1}{2}})\\ &+ \omega_{\overline{q}_{\frac{1}{2}}}(-u_1 + 3ww_1 + \psi_1 \overline{q}_{\frac{1}{2}})\\ &+ \omega_{q_{\frac{3}{2}}}(u) + \omega_{\overline{q}_{\frac{3}{2}}}(-w_1) + \omega_{p_{1,4}}(2\psi_1) + \omega_{p_{1,2}}(\psi_1)\\ &+ \omega_{p_{1,1}}(-\psi_1 - q_{\frac{1}{2}} w_1 - \overline{q}_{\frac{1}{2}} u).\end{aligned} \tag{19}$$

It should be noted that the components are given in the right-module structure .

References

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